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A generalization of the Nataf transformation to distributions with elliptical copula

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ABSTRACT

In the first article [Lebrun R, Dutfoy A. An innovating viewpoint of the isoprobabilistic Nataf transformation with the copula theory within exceedance threshold uncertainty analysis. Probabilistic Structure Engineering Structural Reliability. 2008 [in press]], we showed that the Nataf transformation is a way to model the dependence structure of a random vector by a normal copula, parameterized by its correlation matrix. Following this analysis, we propose an extension of this transformation to any elliptical copula, and give the equivalent of the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM) for this generalization. In particular, we derive the Breitung asymptotic approximation in this new context.

1. Introduction

An isoprobabilistic transformation is one of the most important steps in the use of the First Order Reliability Method (FORM, see [7]), when one needs to compute an approximation of the probability of a rare event, see [3,4]. In the context of an uncertainty treatment analysis, the event E takes the form:

$$E = \{\underline{x} \in \mathbb{R}^n \mid g(\underline{x}) > s\} \tag{1}$$

where $\underline{x}$ is the vector of input parameters defining the state of the studied system, g is the limit state function modeling the system behavior from its state and s the threshold above which the system is considered to be failing. In uncertainty treatment analysis, $\underline{X}$ is a random vector.

The isoprobabilistic transformation T is a diffeomorphism from $supp(\underline{X})$ into $\mathbb{R}^n$, such that the distribution of the random vector $\underline{U} = T(\underline{X})$ has the following properties (see [12]): $\underline{U}$ and $\underline{\underline{R}}\,\underline{U}$ have the same distribution for all rotations $\underline{\underline{R}} \in \mathcal{SO}_n(\mathbb{R})$. Such a transformation exists (see [13,15]), and the most widely used is the Nataf transformation.

In [12] we show that traditional use of the Nataf transformation not only transforms $\underline{X}$ into $\underline{U}$ with such properties, but also completes the probabilistic modeling of $\underline{X}$ such that $\underline{U}$ has the standard n-dimensional normal distribution, namely by assigning to $\underline{X}$ a normal copula.

The objective of this article is threefold: to give a quick introduction to elliptical copulas, to propose a generalization of the Nataf transformation to any random vector $\underline{X}$ which copula is elliptical and not necessarily Gaussian, and to provide an extension of the FORM and SORM approximations to this generalized Nataf transformation. It is a detailed exposition of the results originally presented in [10].

2. Spherical and elliptical distributions

The objective of this section is to give a quick introduction to spherical and elliptical distributions. It is a key step for the presentation of the elliptical copula and the generalized Nataf transformation. The reader will find a much more detailed presentation in [9], as well as the proof of the results we present.

We first define spherical distribution, which is a step towards the definition of elliptical distribution.

Definition 1 (*Spherical Distribution*). A random vector $\underline{X}$ in $\mathbb{R}^n$ has a *spherical distribution* if and only if:

$$\forall \underline{\underline{Q}} \in \mathcal{O}_n(\mathbb{R}), \quad \underline{X} \overset{dist.}{=} \underline{\underline{Q}}\,\underline{X}. \tag{2}$$

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It means that a distribution is spherical if and only if it is invariant by orthogonal transformation. We give two characteristic properties, which could be taken as alternative definitions and are more suited to some proofs.

Proposition 2 (*Stochastic Representation*). *A random vector $\underline{X}$ in $\mathbb{R}^n$ has a spherical distribution if and only if there exists a random variate $R \geq 0$ and a random vector $\underline{U}$ independent of R and uniformly distributed on the hypersphere $\{\underline{x} \in \mathbb{R}^n \mid \underline{x}^t\,\underline{x} = 1\}$, such that:*

$$\underline{X} = R\underline{U}. \tag{3}$$

Using this representation, one can easily sample this kind of distribution.

Proposition 3 (*Characteristic Function*). *A random vector $\underline{X}$ in $\mathbb{R}^n$ has a spherical distribution if and only if its characteristic function $\varphi_{\underline{X}}$ may be written:*

$$\varphi_{\underline{X}}(\underline{u}) = E[e^{\underline{u}^t\underline{X}}] = \psi(\underline{u}^t\,\underline{u}) \tag{4}$$

where ψ is a scalar function, called the characteristic generator of the distribution.

We note $\mathcal{S}_\psi$ the spherical distribution of $\underline{X}$.

This property allows one to prove easily the following classical result: "the only spherical distributions with independent components are the normal distributions with zero mean and scalar covariance matrix $\lambda\underline{\underline{I}}_n$ with $\lambda > 0$". For a demonstration, see e.g. [1].

The function ψ characterizes the *type* of the spherical distribution (e.g. Gaussian, Student etc.), up to a scaling factor: for any constant $c > 0$, $\underline{X}$ and $c\underline{X}$ are of the same type, which means that ψ and $\psi(c^2\,.)$ define the same type of spherical distributions.

If the distribution of $\underline{X}$ is continuous, by inverting its characteristic function, it is easy to see that its probability density function $p_{\underline{X}}$ takes the form:

$$p_{\underline{X}}(\underline{x}) = g(\underline{x}^t\,\underline{x}) \tag{5}$$

where g is a positive scalar function. The function g is called *density generator* of the distribution.

The mean and the covariance of a spherical distribution exist if and only if they exist for the distribution of the associated R (see Eq. (3)). Given that $\mathbb{E}[\underline{U}] = \underline{0}$ and the independence between R and $\underline{U}$, we have:

If $\mathbb{E}[R] < \infty$,

$$\mathbb{E}[\underline{X}] = \mathbb{E}[R\underline{U}] = \mathbb{E}[R]\mathbb{E}[\underline{U}] = \underline{0} \tag{6}$$

If $\mathbb{E}[R^2] < \infty$,

$$\begin{aligned} cov[\underline{X}] &= \mathbb{E}[\underline{X}\,\underline{X}^t] - \mathbb{E}[\underline{X}]\,\mathbb{E}[\underline{X}]^t \\ &= \mathbb{E}[R^2]\mathbb{E}[\underline{U}\,\underline{U}^t] \\ &= \mathbb{E}[R^2]cov[\underline{U}]. \end{aligned} \tag{7}$$

If we take the standard normal distribution $\mathcal{N}(\underline{0}, \underline{\underline{I}}_n)$ for $\underline{X}$, the quantity $\|\underline{X}\|^2 = R^2$ is $\chi^2(n)$-distributed, so $\mathbb{E}[R^2] = n$ and (7) rewrites:

$$\begin{aligned} cov[\underline{X}] &= n\,cov[\underline{U}] = \underline{\underline{I}}_n \\ cov[\underline{U}] &= \frac{1}{n}\underline{\underline{I}}_n. \end{aligned} \tag{8}$$

From (7) and (8) we deduce that for a general spherically distributed $\underline{X}$ such that $\mathbb{E}[R^2] < \infty$:

$$cov[\underline{X}] = \frac{1}{n}\mathbb{E}[R^2]\underline{\underline{I}}_n. \tag{9}$$

We can now define elliptical distribution:

Definition 4 (*Elliptical Distribution*). A random vector $\underline{X}$ in $\mathbb{R}^n$ has an *elliptical distribution* if and only if there exists a deterministic vector $\underline{\mu} \in \mathbb{R}^n$, an n by p deterministic matrix $\underline{\underline{A}}$ and a spherically distributed random vector $\underline{V} \in \mathbb{R}^p$ with $p = rank(\underline{X}) \leq n$ such that:

$$\underline{X} = \underline{\mu} + \underline{\underline{A}}\,\underline{V} \tag{10}$$

where $rank(\underline{X})$ is defined as the dimension of the smallest subspace of $\mathbb{R}^n$ in which $\underline{X}$ takes its values almost surely.

A (possibly degenerated) elliptically distributed random vector is thus the image of a (possibly lower dimensional) spherically distributed random vector by an affine transformation. With this definition, we see that the class of spherical distribution is a special case of the class of elliptical distribution, by taking $\underline{\mu} = \underline{0}$ and $\underline{\underline{A}} \in \mathcal{O}_n(\mathbb{R})$.

Using the stochastic representation of $\underline{V}$, we get:

Proposition 5 (*Stochastic Representation*). *A random vector $\underline{X}$ in $\mathbb{R}^n$ has an elliptical distribution if and only it is possible to find a deterministic vector $\underline{\mu}$, an n by p matrix $\underline{\underline{A}}$ with $p = rank(\underline{X})$, a positive scalar random variate R and a random vector $\underline{U}$ independent of R and uniformly distributed on the unit hypersphere of $\mathbb{R}^p$, such that:*

$$\underline{X} = \underline{\mu} + R\underline{\underline{A}}\,\underline{U}$$

which is the decomposition used for the generation of realizations of an elliptical distribution. In terms of characteristic function, the following result holds.

Proposition 6 (*Characteristic Function*). *A random vector $\underline{X}$ in $\mathbb{R}^n$ has an elliptical distribution if and only if there exists a deterministic vector $\underline{\mu}$ such that the characteristic function of $\underline{X} - \underline{\mu}$ is a scalar function of the quadratic form $\underline{u}^t\,\underline{\underline{\Sigma}}\,\underline{u}$:*

$$\varphi_{\underline{X}-\underline{\mu}}(\underline{u}) = \psi(\underline{u}^t\,\underline{\underline{\Sigma}}\,\underline{u})$$

with $\underline{\underline{\Sigma}}$ a symmetric positive definite matrix of rank p. The matrix $\underline{\underline{\Sigma}}$ is related to the $\underline{\underline{A}}$ of the Proposition 5 through the relation $\underline{\underline{\Sigma}} = \underline{\underline{A}}\,\underline{\underline{A}}^t$.

We note $\mathcal{E}_{\underline{\mu},\underline{\underline{\Sigma}},\psi}$ the elliptical distribution of $\underline{X}$.

If the distribution of $\underline{X} - \underline{\mu}$ is continuous (which implies that $\underline{\underline{\Sigma}}$ is invertible), by inverting its characteristic function, its probability density function $p_{\underline{X}-\underline{\mu}}$ takes the form:

$$p_{\underline{X}-\underline{\mu}}(\underline{x}) = \left(\det\underline{\underline{\Sigma}}\right)^{-1/2} g(\underline{x}^t\,\underline{\underline{\Sigma}}^{-1}\,\underline{x}) \tag{11}$$

where g is a positive scalar function. The function g is called *density generator* of the distribution.

Elliptical distributions share many of the properties of the multivariate normal distribution (which is a special case of elliptical distributions), one of which is the algebra under affine transformation:

Proposition 7 (*Affine Transformation*). *Let $\underline{X}$ in $\mathbb{R}^n$ be a random vector with distribution $\mathcal{E}_{\underline{\mu},\underline{\underline{\Sigma}},\psi}$, $\underline{\underline{A}}$ a deterministic p by n matrix and $\underline{b}$ in $\mathbb{R}^p$ a deterministic vector. The distribution of $\underline{Y} = \underline{\underline{A}}\,\underline{X} + \underline{b}$ is $\mathcal{E}_{\underline{\mu}',\underline{\underline{\Sigma}}',\psi}$, where:*

$$\begin{aligned} \underline{\mu}' &= \underline{a} + \underline{\underline{\Sigma}}\,\underline{\mu} \\ \underline{\underline{\Sigma}}' &= \underline{\underline{A}}\,\underline{\underline{\Sigma}}\,\underline{\underline{A}}^t. \end{aligned} \tag{12}$$

The set of elliptical distributions of a given type is invariant under affine transformation. It follows from this property that any lower dimensional marginal distribution of an elliptical distribution is of the same type as the initial distribution (take for $\underline{\underline{A}}$ the matrix that extracts the needed components and $\underline{b} = \underline{0}$).

Let $\underline{X}$ be a random vector that follow a given elliptical distribution and $\underline{V}$ If the underlying spherical distribution has finite mean and covariance, the elliptical distribution has finite mean and covariance too and we have:

$$\mathbb{E}[\underline{X}] = \mathbb{E}[\underline{\mu} + \underline{\underline{A}}\,\underline{V}] = \underline{\mu} + \underline{\underline{A}}\,\mathbb{E}[\underline{V}] = \underline{\mu} \tag{13}$$

and

$$cov[\underline{X}] = cov[\underline{\mu} + \underline{\underline{A}}\,\underline{V}] = \underline{\underline{A}}\,cov[\underline{V}]\,\underline{\underline{A}}^t = \frac{1}{n}\mathbb{E}[R^2]\underline{\underline{\Sigma}}. \tag{14}$$

The probabilistic distribution of R characterizes the type of elliptical distribution. For example, for a Gaussian distribution of dimension n, R^2 follows a χ_2 distribution with n degrees of freedom.

Remark 8. The expression in Proposition 5 shows that the pair $(R, \underline{\underline{A}})$ is defined up to a multiplicative constant. If $\mathbb{E}[R^2] < \infty$, we will assume that this constant has been chosen such that $\mathbb{E}[R^2] = n$. Then, $\underline{\underline{\Sigma}}$ is exactly the covariance matrix of $\underline{X}$. If $\mathbb{E}[R^2] = \infty$, one can choose the constant such that R^2 has the same median as a $\chi^2(n)$ distribution. Whatever the normalization is, the pair $(R, \underline{\underline{A}})$ is uniquely defined for an elliptically distributed random vector, and the pair $(\underline{\underline{\Sigma}}, \psi)$ is uniquely defined for the associated distribution. We will assume that such a normalization has been made for the remaining of this article.

As $\underline{\underline{\Sigma}}$ is symmetric positive, it can be written in the form $\underline{\underline{\Sigma}} = \underline{\underline{D}}\,\underline{\underline{R}}\,\underline{\underline{D}}$, where $\underline{\underline{D}}$ is the diagonal matrix $\underline{\underline{diag\sigma_i}}$ with $\forall i \in \{1, \ldots, n\}$, $\sigma_i = \sqrt{\Sigma_{ii}} \leq 0$. We note $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n)$. If the distribution has a covariance matrix, with our choice of normalization for ψ, we know that this covariance matrix is equal to $\underline{\underline{\Sigma}}$. The matrix $\underline{\underline{R}}$, such that $R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$, is then its linear correlation matrix. This matrix is always well-defined, even if the distribution has no finite second moment: even in this case, we call it the correlation matrix of the distribution.

To summarize, an elliptical distribution is fully characterized by its *location parameter* $\underline{\mu}$, equal to the mean of the elliptical distribution if it has a finite first moment, its *marginal scale parameter* $\underline{\sigma}$, equal to the vector of standard deviations of the monodimentional marginal distributions if they have finite second moments, its *correlation matrix* $\underline{\underline{R}}$, equal to the linear correlation matrix of the elliptical distribution if it has finite second moments, and its *characteristic generator* ψ, which is a positive scalar function that characterizes the type of the elliptical distribution.

From now, we will denote by $E_{\underline{\mu},\underline{\sigma},\underline{\underline{R}},\psi}$ the cumulative distribution function of $\mathcal{E}_{\underline{\mu},\underline{\sigma},\underline{\underline{R}},\psi}$, corresponding to the previous notation $\mathcal{E}_{\underline{\mu},\underline{\underline{\Sigma}},\psi}$ where $\underline{\underline{\sigma}} = \underline{\underline{D}}\,\underline{\underline{R}}\,\underline{\underline{D}}$.

3. Copula, generic elliptical representative, standard spherical representative

In this section, we introduce the concept of *copula* and give without proof the main theorem related to this concept. For an in-depth introduction to copula, refer to e.g. [6,8,14].

We start by the definition of a copula:

Definition 9. A copula is a function C defined on $[0, 1]^n$ verifying:

(1) for all $\underline{u} \in [0, 1]^n$ with at least one component equal to 0, $C(\underline{u}) = 0$ (C is *grounded*);

(2) C is n-increasing:

$$\sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2}(-1)^{i_1+\cdots+i_n}C(u_{1i_1}, \ldots, u_{ni_n}) \geq 0$$

with $u_{j1} = a_j$ and $u_{j2} = b_j \forall j \in \{1, \ldots, n\}$ and $\underline{a}, \underline{b} \in [0, 1]^n$, $\underline{a} \leq \underline{b}$

(3) For all $\underline{u} \in [0, 1]^n$ with $u_i = 1 \forall i \in \{1, \ldots, n\}, i \neq k$:

$C(\underline{u}) = u_k$.

A copula appears to be the cumulative distribution function (CDF) of a multidimensional distribution with uniform marginal distributions on [0, 1]. The interest of this notion is explained in the following theorem:

Theorem 10 (*SKLAR, 1959*). *Let F be a cumulative density function of dimension n whose marginal distributions are $F_1, \ldots, F_n$. There exists a copula C of dimension n such that for $\underline{x} \in \overline{\mathbb{R}}^n$, we have:*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

If the marginal distributions $F_1, \ldots, F_n$ are continuous, the copula C is unique; otherwise, it is uniquely determined on $\mathrm{Range}(F_1) \times \cdots \times \mathrm{Range}(F_n)$.

In the case of continuous marginal distributions, for all $\underline{u} \in [0, 1]^n$, we have:

$$C(\underline{u}) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n))$$

and

$$p(\underline{x}) = c(F_1(x_1), \ldots, F_n(x_n))\prod_{i=1}^{n}p_i(x_i)$$

where c is the probabilistic density function of C and p_i is the probabilistic density function of the i-th marginal distribution of $\underline{X}$.

We see that a multidimensional probability distribution, which is completely described by its CDF F, is made of a collection of marginal distributions, given by the marginal CDF F_i, and a copula C which describes *exactly* the dependence structure between the components of the distribution.

We emphasize that the copula of a distribution is a *multivariate function* and as such it cannot be summarized by a set of scalars without further hypothesis. Some of the hazards of such an attempt are exposed in [12].

Definition 11 (*Elliptical Copula*). An *elliptical copula* $C^E_{\underline{\underline{R}},\psi}$ is the copula of an elliptical distribution $\mathcal{E}_{\underline{\mu},\underline{\sigma},\underline{\underline{R}},\psi}$.

Remark 12. Thanks to the normalization presented in Remark 8, the mapping $(\underline{\underline{R}}, \psi) \mapsto C^E_{\underline{\underline{R}},\psi}$ is one to one. The *type* of the copula is given by ψ and its *shape* by $\underline{\underline{R}}$.

In general, the copula $C^E_{\underline{\underline{R}},\psi}$ is *not* the cumulative distribution function of an elliptical distribution itself.

It is clear that mapping between elliptical distributions and elliptical copulas is not one-to-one. Let $\mathcal{R}$ be the equivalence relation between elliptical distributions: $\mathcal{E}_{\underline{\mu}_1,\underline{\sigma}_1,\underline{\underline{R}}_1,\psi_1} \overset{\mathcal{R}}{\equiv} \mathcal{E}_{\underline{\mu}_2,\underline{\sigma}_2,\underline{\underline{R}}_2,\psi_2}$ if and only if $\mathcal{E}_{\underline{\mu}_1,\underline{\sigma}_1,\underline{\underline{R}}_1,\psi_1}$ and $\mathcal{E}_{\underline{\mu}_2,\underline{\sigma}_2,\underline{\underline{R}}_2,\psi_2}$ share the same copula $C^E_{\underline{\underline{R}},\psi}$. From Remark 12, this relation reads $(\underline{\underline{R}}_1, \psi_1) = (\underline{\underline{R}}_2, \psi_2)$.

We introduce the notion of generic representative to distinguish one particular elliptical distribution of each class of equivalence:

Definition 13 (*Generic Elliptical Representative*). The *generic elliptical representative* of an elliptical distribution family $\mathcal{E}_{\underline{\mu},\underline{\sigma},\underline{\underline{R}},\psi}$

Fig. 1. Graph showing the relations between the several kinds of elliptical and spherical distributions (oval nodes) introduced in the article, and how general distributions with elliptical copulas (rectangular nodes) are linked to these distributions through bijection between elliptical copulas and generic elliptical representatives. The labels on the links are related to what is extracted from the left-hand side to go to the right-hand side. For example, extraction of the dependence structure of a general distribution with an elliptical copula leads to its copula.

through the equivalence relation $\mathcal{R}$ is the elliptical distribution whose cumulative distribution function is $E_{\underline{0},\underline{1},\underline{\underline{R}},\psi}$.

All the other members of the equivalence class differ only by their location parameter $\underline{\mu}$ and their marginal scale parameter $\underline{\sigma}$.

We introduce a last kind of elliptical distribution that allows one to focus on the type of a elliptical distribution, throwing away the shape information.

Definition 14 (*Standard Spherical Representative*). The *standard spherical representative* of an elliptical distribution family $\mathcal{E}_{\underline{\mu},\underline{\sigma},\underline{\underline{R}},\psi}$ is $\mathcal{S}_{\psi}$, the spherical distribution whose cumulative distribution function is $S_{\psi} = E_{\underline{0},\underline{1},\underline{\underline{I}}_n,\psi}$.

It is the only member of the elliptical family which is both spherical and with null location parameter and unit marginal scale parameter.

Definition 15 (*Distribution with Elliptical Copula*). The family of distributions with marginal cumulative distribution functions are $F_1, \ldots, F_n$ and any elliptical copula $C^E_{\underline{\underline{R}},\psi}$ is denoted by $\mathcal{D}_{F_1,\ldots,F_n,C^E_{\underline{\underline{R}},\psi}}$. The cumulative distribution function of this distribution is noted $D_{F_1,\ldots,F_n,C^E_{\underline{\underline{R}},\psi}}$.

The relationship between the different kind of elliptical distributions is depicted in Fig. 1. We also show how general distributions with elliptical copulas interact with these distributions.

4. Generalized Nataf transformation

The usual Nataf transformation [13] has already been analysed in the light of the copula theory, see [12]. It is shown that this transformation is well defined if and only if the copula of the transformed random vector is normal, and some of the consequences of such an hypothesis are presented in [11,12,10]. Here, we propose a generalization of this transformation to a distribution with any elliptical copula. In this section, the random vector $\underline{X}$ is supposed to be continuous and with full rank. We also suppose that its cumulative marginal distribution functions are strictly increasing (so they are bijective) and that the matrix $\underline{\underline{R}}$ of its elliptical copula is symmetric positive definite.

Following the analysis given in [12], the usual Nataf transformation can be defined by:

Definition 16 (*Usual Nataf Transformation*). Let $\underline{X}$ in $\mathbb{R}^n$ be a continuous random vector following the distribution $D_{F_1,\ldots,F_n,C^E_{\underline{\underline{R}},\psi_{\mathcal{N}}}}$, where $\psi_{\mathcal{N}}$ is the generator of the normal distributions. The *Nataf transformation* T_{Nataf} is defined by:

$$\underline{u} = T_{Nataf}(\underline{x}) = T_3 \circ T_2 \circ T_1(\underline{x}) \tag{15}$$

where the three transformations T_1, T_2 and T_3 are given by:

$$\begin{aligned} &T_1 : \mathbb{R}^n \to \mathbb{R}^n \\ &\underline{x} \mapsto \underline{w} = (F_1(x_1), \ldots, F_n(x_n))^t \\ &T_2 : \mathbb{R}^n \to \mathbb{R}^n \\ &\underline{w} \mapsto \underline{v} = (\Phi^{-1}(w_1), \ldots, \Phi^{-1}(w_n))^t \\ &T_3 : \mathbb{R}^n \to \mathbb{R}^n \\ &\underline{v} \mapsto \underline{u} = \underline{\underline{\Gamma}}\, \underline{v} \end{aligned} \tag{16}$$

where Φ is the cumulative distribution function of the standard 1-dimensional normal distribution and $\underline{\underline{\Gamma}} = \underline{\underline{L}}^{-1}$, where $\underline{\underline{L}}$ is the Cholesky factor of $\underline{\underline{R}}$, defined as the unique lower triangular matrix such as $\underline{\underline{R}} = \underline{\underline{L}}\,\underline{\underline{L}}^t$. The hypothesis made on the distribution of $\underline{X}$ insure that $\underline{\underline{L}}$ is well defined and that T_1 is bijective.

It has been shown in [12] that the distribution of $\underline{U} = T_{Nataf}(\underline{X})$ is the standard n-dimensional normal distribution, namely the standard spherical representative associated to $C^E_{\underline{\underline{R}},\psi_{\mathcal{N}}}$. The transformation T_1 maps $\underline{X}$ into a random vector $\underline{W}$ whose distribution is the normal copula $C^E_{\underline{\underline{R}},\psi_{\mathcal{N}}}$, the transformation T_2 maps $\underline{W}$ into a random vector $\underline{V}$ whose distribution is the generic normal representative associated to $C^E_{\underline{\underline{R}},\psi_{\mathcal{N}}}$ and T_3 maps $\underline{V}$ into a random vector whose distribution is the standard normal representative associated with $C^E_{\underline{\underline{R}},\psi_{\mathcal{N}}}$. The $\underline{U}$-space is called the *standard space* whereas the $\underline{X}$-space is called the *physical space*. With this point of view, a natural generalization of the Nataf transformation is the following:

Definition 17 (*Generalized Nataf Transformation*). Let $\underline{X}$ in $\mathbb{R}^n$ be a continuous random vector following the distribution $D_{F_1,\ldots,F_n,C^E_{\underline{\underline{R}},\psi}}$. The *generalized Nataf transformation* T^{gen}_{Nataf} is defined by:

$$\underline{u} = T^{gen}_{Nataf}(\underline{x}) = T^{gen}_3 \circ T^{gen}_2 \circ T_1(\underline{x}) \tag{17}$$

where the three transformations T_1, T^{gen}_2 and T^{gen}_3 are given by:

$$\begin{aligned} &T_1 : \mathbb{R}^n \to \mathbb{R}^n \\ &\underline{x} \mapsto \underline{w} = (F_1(x_1), \ldots, F_n(x_n))^t \\ &T^{gen}_2 : \mathbb{R}^n \to \mathbb{R}^n \\ &\underline{w} \mapsto \underline{v} = (E^{-1}(w_1), \ldots, E^{-1}(w_n))^t \\ &T_3 : \mathbb{R}^n \to \mathbb{R}^n \\ &\underline{v} \mapsto \underline{u} = \underline{\underline{\Gamma}}\, \underline{v} \end{aligned} \tag{18}$$

where E is the cumulative distribution function of standard 1-dimensional elliptical distribution with characteristic generator ψ and $\underline{\underline{\Gamma}}$ is the inverse of the Cholesky factor of $\underline{\underline{R}}$.

This transformation differs from the usual one by its second step, which is modified such that the distribution of $\underline{W} = T_2^{gen} \circ T_1(\underline{X})$ is the generic elliptical representative associated with the copula of $\underline{X}$. The step T_3 maps this distribution into its standard representative, following exactly the same algebra as the normal copula. In the special case where the distribution of $\underline{X}$ is already elliptical, with cumulative distribution function $E_{\underline{\mu},\underline{\sigma},\underline{\underline{R}},\psi}$, the generalized Nataf transformation is an affine transformation: the transformation $T_2^{gen} \circ T_1$ maps the elliptical distribution into its generic representative, which is an affine transformation of each component, and the transformation T_3 is linear, thus $T_{Nataf}^{gen} = T_3 \circ T_2^{gen} \circ T_1$ is affine. More precisely, if we note $\underline{V} = T_2^{gen} \circ T_1(\underline{X})$, we have $V_i = (X_i - \mu_i)/\sigma_i$ for all $i \in \{1, \ldots, n\}$. The generalized Nataf transformation can then be expressed in this case as:

$$\underline{U} = T_{Nataf}^{gen}(\underline{X}) = \underline{\underline{S}}(\underline{X} - \underline{\mu}) \qquad (19)$$

where $\underline{\underline{S}}$ is the inverse of the Cholesky factor of $\underline{\underline{\Sigma}} = \underline{\underline{D}}\,\underline{\underline{R}}\,\underline{\underline{D}}$.

5. FORM and SORM approximations

Once the Nataf transformation has been extended to elliptical distributions, it is necessary to provide an extension of the FORM and SORM approximations to make the evaluation of the probability of exceedance possible.

Given a numerical model $f : \mathbb{R}^n \to \mathbb{R}$ with suitable properties of differentiability, and a threshold $s \in \mathbb{R}$, the evaluation of the probability:

$$P_f = \mathbb{P}[f(\underline{X}) > s] = \int_{\mathbb{R}^n} \mathbf{1}_{f(\underline{x})>s}\, p_{\underline{X}}(\underline{x})\, \mathrm{d}\underline{x} \qquad (20)$$

where $p_{\underline{X}}$ is the probability density function of $\underline{X}$, can be transformed in the evaluation of the probability:

$$P_f = \mathbb{P}[G(\underline{U}) > s] = \int_{\mathbb{R}^n} \mathbf{1}_{G(\underline{u})>s}\, p_{\underline{U}}(\underline{u})\, \mathrm{d}\underline{u} \qquad (21)$$

where T is a C^1-diffeomorphism called an *iso-probabilistic transformation*, $p_{\underline{U}}$ is the probability density function of $\underline{U} = T(\underline{X})$ and $G = f \circ T^{-1}$. The vector $\underline{U}$ is said to be in the *standard space*, whereas $\underline{X}$ is in the *physical space*.

The interest of such a transformation is that if the density function $p_{\underline{U}}$ is a function of $\|\underline{u}\|$, which means that $\underline{U}$ has a spherical distribution. The generalized Nataf transformation is such a transformation, as far as $\underline{X}$ has an elliptical copula, which is the case we consider here.

If $p_{\underline{U}}$ is also a rapidly decreasing function of $\|\underline{u}\|$ in the failure domain $\mathcal{D} = \{\underline{u} \mid G(\underline{u}) > s\}$, the integral in (21) can be approximated by an integral of $p_{\underline{U}}$ over an approximate domain $\tilde{\mathcal{D}}$ close to $\mathcal{D}$. This domain is usually obtained by an approximation of the boundary of $\mathcal{D}$ at the vicinity of the point $\underline{u}^* \in \mathcal{D}$ with minimal norm, i.e. with maximum density (such a point is called the *design point*). A linear approximation of $\mathcal{D}$ at $\underline{u}^*$ leads to the FORM approximation, whereas a quadratic one leads to the SORM approximation.

The distribution invariance by orthogonal transformation of $\underline{U}$ allows us to suppose that the design point $\underline{u}^*$ has components $(0, \ldots, 0, \beta)^t$, where $\beta = \|\underline{u}^*\| \geq 0$, see Fig. 2.

In the case of a linear approximation, the hyperplane tangent to $\mathcal{D}$ at $\underline{u}^*$ is normal to $\underline{u}^*$, thus the generalized FORM approximation of the probability of failure is:

$$P_{FORM}^{gen} = \mathbb{P}[U_1 > \beta] = 1 - E(\beta) = E(-\beta) \qquad (22)$$

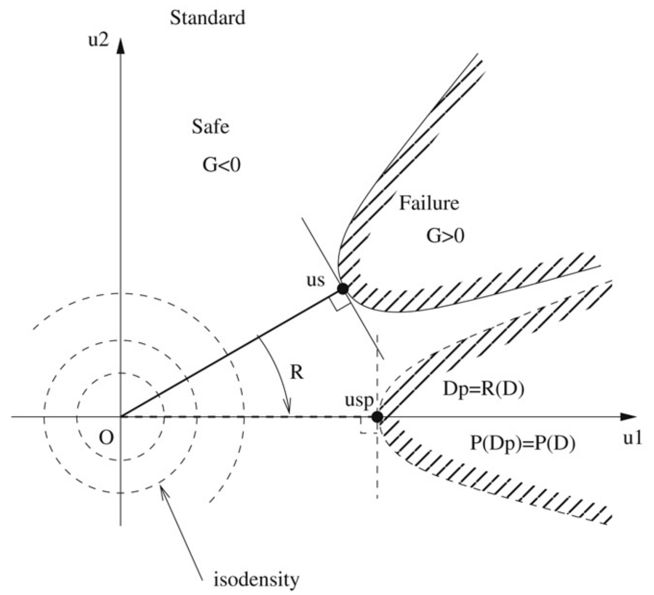

Fig. 2. Rotational invariance after the application of the generalized Nataf transformation. The rotational invariance of $\mathcal{E}_{\underline{0},\underline{1},\underline{\underline{I}},\psi}$ allows one to focus on the situation depicted in dashed form without loss of generality, thanks to the rotation $\underline{\underline{R}}$ that maps a general failure domain $\mathcal{D}$ to a domain $\mathcal{D}'$ for which the design point $\underline{u}^{*\prime}$ is supported by the last axis.

where E is the cumulative distribution function of the 1-D standard elliptical representative of the same type than the copula of $\underline{X}$.

For the generalized SORM approximation, more work is required. As in the case of a normal copula (the usual Nataf transformation), the expression of the probability of failure has no simple analytical formulation. The generalization of Tvedt's exact formula [16,17] does not seem to extend easily to the more general context we study here, as its proof relies on the independence of the components of the standard spherical representative, which occurs only in the Gaussian case. But it is possible to generalize Breitung's formula [2], which is an asymptotically exact formula when $\beta \to \infty$ that is based on the application of the saddle-point method to the evaluation of the integral in (21). We recall here the general result demonstrated in [2]:

Proposition 18 (*Breitung's Approximation, General Case*). *If we rewrite the density function $p_{\underline{U}}$ in the form:*

$$p_{\underline{U}}(\underline{u}) = \exp(l(\underline{u})) \qquad (23)$$

and if we suppose that the design point $\underline{u}^*$ is unique, then we have:

$$P_f \overset{\beta\to\infty}{=} (2\pi)^{\frac{n-1}{2}} \frac{p_{\underline{U}}(\underline{u}^*)}{\sqrt{|J|}} + o(1) \qquad (24)$$

where:

$$J = \underline{\nabla l}(\underline{u}^*)^t\, \underline{\underline{C}}(\underline{u}^*)\, \underline{\nabla l}(\underline{u}^*) \qquad (25)$$

$$\underline{\underline{C}}(\underline{u}^*) = \text{co-factor matrix of } \underline{\underline{H}}(\underline{u}^*) \qquad (26)$$

$$\underline{\underline{H}}(\underline{u}^*) = \left(\underline{\underline{\nabla^2 l}}(\underline{u}^*) - \lambda \underline{\underline{\nabla^2 G}}(\underline{u}^*)\right) \qquad (27)$$

$$\lambda = \frac{\|\underline{\nabla l}(\underline{u}^*)\|}{\|\underline{\nabla G}(\underline{u}^*)\|}. \qquad (28)$$

The gaussian case is also examined in [2], which leads to the following results:

Proposition 19 (*Breitung's Approximation, Gaussian Case*). *Formulas from (25) to (28) rewrite:*

$$\beta = \|\underline{u}^*\|, \qquad \lambda = \beta, \qquad \underline{\nabla l}(\underline{u}^*) = -\underline{u}^*,$$
$$\underline{\underline{H}}(\underline{u}^*) = -\underline{\underline{I}}_n - \beta \underline{\underline{\nabla^2 G}}(\underline{u}^*),$$
$$J = \beta^2 \prod_{i=1}^{n-1}(1+\beta\kappa_i)$$
$$\text{where } \kappa_i = \text{ main curvatures of } \{G(\underline{u}) = 0\} \text{ at } \underline{u}^*$$

and we have:

$$P_f \overset{\beta\to\infty}{=} \frac{1}{\beta\sqrt{2\pi}} \exp(-\beta^2/2) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1+\beta\kappa_i}} + o(1) \tag{29}$$

$$\overset{\beta\to\infty}{=} \Phi(-\beta) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1+\beta\kappa_i}} + o(1). \tag{30}$$

We obtained the last relation thanks to Mill's ratio:

$$\Phi(-x) \overset{x\to\infty}{=} \frac{1}{x\sqrt{2\pi}} \exp(-x^2/2) + o(1) \tag{31}$$

and it is the usual form of what is called Breitung's formula in the reliability literature (sometimes with the opposite sign for curvatures, depending on the chosen convention for their sign). With this formulation, one can see Breitung's formula as a geometric correction of the FORM approximation through the factor $\prod_{i=1}^{n-1} \frac{1}{\sqrt{1+\beta\kappa_i}}$.

In the case of a general elliptical copula $C^E_{\underline{\underline{R}},\psi}$, we have:

$$l(\underline{u}) = \log(g(\|\underline{u}\|^2)) \tag{32}$$

where g is the density generator of the standard elliptical distribution associated to $C^E_{\underline{\underline{R}},\psi}$, see (11).

Without loss of generality, due to the rotational invariance of (21) and the fact that the limit state function G plays a role only through its derivatives up to the second order in the formulas, we can suppose that $\underline{u}^*$ is positively proportional to $\underline{e}_n$ and that G is a linear function of u_n and a quadratic function of the other components of $\underline{u}$:

$$\underline{u}^* = \beta \underline{e}_n, \qquad \beta \geq 0 \tag{33}$$

$$G(\underline{u}) = \beta - \underline{e}_n^t \underline{u} + \frac{1}{2} \underline{u}^t \underline{\underline{A}}\, \underline{u} \tag{34}$$

$$\underline{\underline{A}} = \left(\begin{array}{c|c} \tilde{\underline{\underline{A}}} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$\text{with } \tilde{\underline{\underline{A}}} \text{ a symmetric } (n-1)\times(n-1) \text{ matrix.} \tag{35}$$

In this form, the main curvatures of $\{G(\underline{u}) = 0\}$ at $\underline{u}^*$ are exactly the eigenvalues $(\kappa_i)_{i=1,\dots,n-1}$ of $\tilde{\underline{\underline{A}}}$. The gradient and the hessian of G at $\underline{u}^*$ are:

$$\underline{\nabla G}(\underline{u}^*) = -\underline{e}_n + \underline{\underline{A}}\, \underline{u}^* = -\underline{e}_n \text{ as } \underline{\underline{A}}\, \underline{e}_n = \underline{0} \tag{36}$$

$$\underline{\underline{\nabla^2 G}}(\underline{u}^*) = \underline{\underline{A}}. \tag{37}$$

We evaluate the gradient and the hessian of l at the design point $\underline{u}^*$:

$$\underline{\nabla l}(\underline{u}^*) = 2\frac{g'(\beta^2)}{g(\beta^2)} \underline{u}^* = 2\beta \frac{g'(\beta^2)}{g(\beta^2)} \underline{e}_n \tag{38}$$

$$\underline{\underline{\nabla^2 l}}(\underline{u}^*) = 2\left[2\left(\frac{g''(\beta^2)}{g(\beta^2)} - \left(\frac{g'(\beta^2)}{g(\beta^2)}\right)^2\right) \underline{e}_n \underline{e}_n^t + \frac{g'(\beta^2)}{g(\beta^2)} \underline{\underline{I}}_n\right] \tag{39}$$

where g' and g'' denote the first and the second derivatives of g.

Thus, Eq. (28) reads:

$$\lambda = 2\beta \frac{|g'(\beta^2)|}{g(\beta^2)} = -2\beta \frac{g'(\beta^2)}{g(\beta^2)} \tag{40}$$

because the design point is the point of the failure domain where the probability density function is maximum, so $g'(\beta^2) \leq 0$.

Eq. (25) reads:

$$J = \left(2\beta \frac{g'(\beta^2)}{g(\beta^2)}\right)^2 \underline{e}_n^t C(\underline{u}^*) \underline{e}_n = \left(2\beta \frac{g'(\beta^2)}{g(\beta^2)}\right)^2 C_{nn}(\underline{u}^*) \tag{41}$$

where $C_{nn}(\underline{u}^*)$ is the cofactor of $H_{nn}(\underline{u}^*)$, i.e. the determinant of its $(n-1)\times(n-1)$ upper left block. We have:

$$C_{nn}(\underline{u}^*) = \left(2\frac{g'(\beta^2)}{g(\beta^2)}\right)^{n-1} \det\left(\underline{\underline{I}}_{n-1} + \beta\tilde{\underline{\underline{A}}}\right)$$
$$= \left(2\frac{g'(\beta^2)}{g(\beta^2)}\right)^{n-1} \prod_{i=1}^{n-1}(1+\beta\kappa_i). \tag{42}$$

Using (24), (41) and (42) we get:

$$P_f \overset{\beta\to\infty}{=} \frac{1}{\beta}(2\pi)^{\frac{n-1}{2}} \left(-\frac{g(\beta^2)}{2g'(\beta^2)}\right)^{\frac{n+1}{2}}$$
$$\times g(\beta^2) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1+\beta\kappa_i}} + o(1) \tag{43}$$

which is Breitung's formula extended to the general elliptical case. This formula has the same form as (29): it is a product between a quantity only related to the distribution (its density generator) and a quantity only related to the geometry of the failure domain $\mathcal{D}$. To see if there is an equivalent of Mill's ratio in the general elliptical case, consider the case $\underline{\underline{A}} = \underline{\underline{0}}$: all the κ_i are 0, and the geometric factor reduces to 1. One the other hand, in this case the failure domain is exactly an half-space and the exact value of the associated probability is given by $P_f = E(-\beta)$. We conclude that:

Proposition 20 (*Mill's Ratio for a General Spherical Distribution*). *Let g be the density generator of an arbitrary n-dimensional spherical distribution and E be its unidimensional marginal CDF. Then we have the approximation:*

$$E(-\beta) \overset{\beta\to\infty}{=} \frac{1}{\beta}(2\pi)^{\frac{n-1}{2}} \left(-\frac{g(\beta^2)}{2g'(\beta^2)}\right)^{\frac{n+1}{2}} g(\beta^2) + o(1). \tag{44}$$

Using this relation, the generalized Breitung's formula can be written:

$$P_f \overset{\beta\to\infty}{=} E(-\beta) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1+\beta\kappa_i}} + o(1) \tag{45}$$

which is exactly the form taken by the usual Breitung's formula in the gaussian case, see (30). The same remark applies here: Breitung's formula is a geometric correction of the FORM approximation, with the same corrective factor as in the gaussian case.

6. Conclusion

In this article, we gave a quick introduction to copula and elliptical distribution theory, and we proposed a generalization of the Nataf transformation to any distribution with elliptical copula. In order to make an effective use of this generalized transformation, we derived the associated FORM and SORM/Breitung approximation of the probability of failure. These approximations appear to be very natural extensions of the Gaussian case associated with the usual Nataf transformation.

It is worth mentioning the new OpenTURNS software (Open Source Treatment of Uncertainties, Risks aNd Statistics): this is the result of a joint research project between EDF R&D, EADS Innovation Works and PhiMECA, in which all the results presented here have been implemented. See the website www.openturns.org or [5] for further information on the software.

References

[1] Arnold SF, Lynch J. On Ali's characterization of the spherical normal distribution. Journal of Royal Statistical Society 1982;44(1):49–51.
[2] Breitung K. Asymptotic approximation for multinormal integrals. Journal of Engineering Mechanics, ASCE 1984;110(3):357–66.
[3] Der Kiureghian A, Liu PL. Structural reliability under incomplete probabilistic information. Journal of Engineering Mechanics 1986;112(1):85–104.
[4] Der Kiureghian A, Liu PL. Multivariate distribution models with prescribed marginals and covariances. Probabilistic Engineering Mechanics 1986;1(2): 105–12.
[5] Dutfoy A, Dutka-Malen I, Lebrun R, Conty R. et al. Open TURNS. An Open source initiative to Treat Uncertainties. Risks'N Statistics in a structured industrial approach. In: ESREL 2007 Conference.
[6] Embrechts P, Lindskog F, McNeil A. Modelling dependance with copulas. ETH preprint. 2001.
[7] Hasofer AM, Lind NC. An exact and invariant first order reliability format. Journal of Engineering Mechanics 1974;100:111–21.
[8] Joe H. Multivariate models and dependance concepts. Chapman & Hall; 1997.
[9] Kotz K-T, Fang S, Ng K-W. Symmetric multivariate and related distributions. Chapman & Hall; 1987.
[10] Lebrun R. Modelling dependence with copulas in reliability analysis, a new approach to the FORM and SORM methods. In: Proceedings of the eights international conference on probabilistic safety assessment and management. PSAM 8, New Orleans (United States): ASME; 2004.
[11] Lebrun R, Dutfoy A. Modélisation de la dépendance stochastique par la théorie des copules: une généralisation de la transformation de Nataf. In: Proceedings of the 18th Congrès Français de Mécanique. 2007.
[12] Lebrun R, Dutfoy A. An innovating viewpoint of the isoprobabilistic Nataf transformation with the copula theory within exceedance threshold uncertainty analysis. Probabilistic Structure Engineering Structural Reliability. 2008 [in press].
[13] Nataf A. Détermination des distributions de probabilités dont les marges sont données. Comptes Rendus de l'Académie des Sciences; 225: 42–3.
[14] Nelsen R. An introduction to copulas. Springer; 1999.
[15] Rosenblatt M. Remarks on a multivariate transformation. The Annals of Mathematical Statistics; 23(3): 470–72.
[16] Tvedt L. Second order reliability by an exact integral. In: Thoft-Christensen, editor. Proceedings of the IFIP working conference reliability and optimization of structural systems. 1988. p. 377–84.
[17] Tvedt L. Distribution of quadratic forms in normal spaces — application to structural reliability. Structural Safety 1988;3:143–66.